Reflections on "Diagrams in Mathematical Education"

Erick Smith
University of Illinois at Chicago

Dennis Lomas argues that diagrams play a vital and valid role in mathematical reasoning. I like what he is trying to do but am troubled by how he is going about it. He makes many questionable assumptions which tend to undermine the holistic value of the essay, even though they support a valid, if unnecessarily weak, conclusion.

The main problem essentially lies in his starting position, which seems to be that proof is unproblematic in the community of professional mathematicians, that mathematicians agree on the nature and centrality of formal proofs in mathematics, and that there is no distinction between mathematical reasoning and mathematical proofs. These assumptions simply belie almost everything that has been written from Godel to Lakatos, from Wittgenstein to Ernest, and from Hana to Henderson. Godel and Lakatos undermined the idea that proof can be infallible and independent of human thought and social processes.[1] Wittgenstein and Ernest have described linguistic communication and the social nature of acceptance of a proof.[2] Hana describes the secondary, almost auxiliary role that formal proofs play in the work of most mathematicians and explains the lack of any widespread criteria for what constitutes a valid proof within the mathematical community. For Hana, there is a huge gulf between "formal proof," which is relatively unimportant, and "mathematical reasoning," which is central to the work of all mathematicians.[3] Henderson goes a step further, arguing that is not in the nature of the form of the proof itself, but rather in the meaning of the individual knower, that a proof achieves validity.[4]

Few mathematicians would argue against the role of diagrams in everyday mathematical reasoning, and reading any mathematical journal provides evidence that diagrams are legitimate components of the mathematical arguments which are at the heart of communication among mathematicians. Thus, it seems quite unnecessary to argue for the legitimacy of diagrams in the everyday activities of mathematicians. Thus, even before turning to the specific issue of diagrams in formal proofs, it seems that Lomas's primary issue can be resolved. There is a strong consensus in the mathematics education community that mathematical reasoning, sense-making, and communication should be central in the K-12 curriculum, much as they are in the practice of mathematicians. A corollary is that if formal proofs are to have any role at all in K-12 mathematics, it should be as a tool to support mathematical reasoning, not the other way around, as is so often assumed.[5] Thus the question is not whether diagrams are a legitimate part of K-12 mathematics, but whether it would even be possible to have a meaningful mathematics curriculum without diagrams as a central component.

Diagrams and Formal Proofs

Since there is no widespread agreement about what counts as a formal proof, one might start with the criterion that it is a proof which only uses symbols and the rules for manipulating them. Although such a definition would exclude diagrams, it raises the question of whether there is any validity to such a restriction, other than historical capriciousness. The first question one might ask is about the nature of the distinction between diagrams and symbols. For Lomas, a symbol seems to represent a general class of mathematical objects, but it is not clear whether he attributes a similar role to diagrams. In the Socrates example, he tends to treat the square in the sand as an actual instance of one particular square. Viewing it in this way validates the necessity of a generalizing inference, such as Lomas gives near the beginning of his essay: "(INF) This particular square is like any other except for size, location, and orientation; the technique for doubling this square does not depend on its size location, or, orientation; therefore." However, if one views a diagram as a representation of a class of objects, such an inference is not necessary. It is not at all clear that most people do think of diagrams as specific examples of a class. In the Socrates example, the "square" in the sand is, of course, not a square. Rather, each actor must take it to be a square. In doing so, it becomes a representation of a square. But in seeing it as a representation of a square, there is no indication that either Socrates or Meno was focusing on its size or orientation — that is, it was not just an example of a particular square for them, but a representation of the class of squares. Squares, as all mathematical objects, are in the minds of the players, not in the physical world. As a representation of "squares," the figure in the sand becomes a tool for thought, a tool that assists one in reasoning about squares. Of course, one must ultimately examine that reasoning process to determine what class of figures are actually compatible with that process, specifically to determine whether the class of squares is compatible. This was, one could argue, just what Socrates was doing when he asked Meno: "And such a figure could be larger or smaller?"

Thus, even if we take a diagram to be a representation of a class of mathematical objects, we must check either during or after the reasoning process to be sure that that process applies to all squares or, if not, what figures it does apply to. However, rather than making diagrams distinct from symbols, this process is *exactly* what we do when arguing with symbols. For example, when arguing about numbers, a division by x - 1 requires us to remove the case of x = 1 from our conclusions, including division by 2 in an argument about integers requires us to limit our conclusion to even integers. And this process extends to more formal proofs. For example, how do we know that 2 + 2 = 4? Mathematicians, in general, would reject as a proof the example of two apples plus two apples, specifically because it appears to be only an example. However, in discussing a formal set-theoretic proof that 2 + 2 = 4, mathematician, J.L. Mackie, claims: "The conclusion that issues from it has exactly the same status as that issuing from the apple proof. That is to say: it depends on an instance of 2 + 2 = 4. We use the result that 2 + 2 = 4 in order to select, order, apprehend and arrange the symbols of the proof."[6] Mackie argues that we accept the proof because we defined our sets for which we already knew the result would hold. Taking Mackie literally would imply that every proof is nothing more than an example. However, we might view either proof as representational of a class of objects for which the conclusion holds. Either proof (sets or apples) is, in fact, valid for all those classes of objects for which 2 + 2 = 4, and is not valid for cases where

2 + 2 does not equal 4 (for example, the metaphorical case of adding apples and oranges). Thus generically, there is very little difference between the roles of symbols and diagrams (and objects) in proof. Both can become representations for classes of mathematical objects which are mental entities. As representations, they play the role of tools to assist thinking about these classes. Both require a careful coordination between the process of reasoning and the class of objects being examined.

Geometric Proof

This does not mean that all representations are equal. Symbols are particularly useful for representing a number, for a shorthand designation of components of a diagram, to represent more complex and higher dimensional figures which cannot easily be represented by a diagram, and to represent the measure of a geometrical object. In the latter two cases, however, there is often a loss. When a symbol represents a geometric object, it becomes a less valuable tool for mathematical reasoning because it has less connection to the geometric objects under consideration.

In Lomas's proof of the Pythagorean theorem, he uses symbols to represent the measure of geometric objects. In doing so, his proof has little to do with geometric reasoning, and, in fact, alters the meaning of this theorem. In figure 1, the Pythagorean theorem states that for any right triangle, 1, the area of the square, A, on the hypotenuse is equal to the combined areas of the squares, B and C, on the two legs. The diagram Lomas uses is essentially that of figure one, where one first creates three copies of triangle 1, then arranges them so as to create square S, which contains the four congruent triangles and square A.

Lomas concludes the proof by using "a" as the measure of the short leg, "b" as the measure of the long leg, and "a+b" as the measure of the side of square S. From this he uses algebraic manipulation to prove his conclusion. Whereas Lomas and others take the position that the algebraic component is the substance of the proof which provides its ultimate validity, I would take just the opposite position. Whenever one takes the measure of a geometric object and then uses numeric reasoning to reach a geometric conclusion, one is using derived or secondary properties which obscure the original geometric argument. One has to do a mental adjustment from focusing on the equivalence of the areas of geometric figures to focusing on the equivalence of the measures of geometric areas. In this case, as in most geometric arguments, such a transformation not only weakens the geometric understanding, but is unnecessary. Figure two shows a simple method of obtaining the desired result without destroying the integrity of the geometric argument. Sliding triangle 4 to the top right of square S while sliding triangles 1 and 3 to the bottom left allows one to see square A dissected into squares B and C without resorting to measurement.

Although I agree with Lomas that language is an important part of reasoning, it is often not the only means of communicating an argument. Others include pictures, body movement, physical contact, expression, and so on. In fact, if an individual understood the goal of showing that the area of square A is equivalent to the combined areas of squares B and C, it is quite likely that she could come to understand the "proof" represented in figures 1 and 2, even if she shared no common language with her teacher. On the other hand, sharing a common language is certainly no guarantee that mathematical communication takes place. A likely example is the case of Socrates and Meno. Despite the fact that they share a common language, it is not at all clear what it is that Meno does come to understand. It seems quite possible that Meno has learned very little about geometry, but is quite adept at figuring out the appropriate way to respond to Socrates' questions (a skill many students learn). One might then argue that the importance of the shared language and culture is more to allow Meno to infer what response Socrates is seeking than to help him understand properties of squares.

Despite these criticisms, I applaud the direction Lomas is taking. For those still ensconced in one particular traditional and narrow view of mathematics and the role of proof, his arguments are valid and will hopefully spread some enlightenment. It would have been helpful, however, if he had situated his arguments more within a framework compatible with the mainstream thought of most mathematicians and mathematics educators.

MY APPRECIATION TO David Henderson, Cornell University, for his helpful comments on this article.

---

1. I. Lakatos, *Proofs and Refutations: The Logic of Mathematical Discovery* (Cambridge: Cambridge University Press, 1976).

2. Ludwig Wittgensten, *Remarks on the Foundations of Mathematics* (Cambridge: MIT Press, 1978) and P. Ernest, "The Dialogical Nature of Mathematics," in *Mathematics, Education and Philosophy: An International Perspective* , ed. P. Ernest (London: Falmer Press, 1994), 33-48.

3. G. Hana, *Rigorous Proof in Mathematics Education* (Toronto: Ontario Institute for Studies in Education, 1983), Curriculum Series #48.

4. D. Henderson, "Proof as a Convincing Argument that Answers — Why?" in *Seventh International Congress of Mathematics Education*, Quebec, Canada, 17-21 August 1992.

5. In addition, Hana suggests that formal proofs could be a topic covered in advanced secondary mathematics courses.

6. D. Bloor, "What Can the Sociologist of Knowledge Say About 2 + 2 = 4?" in Ernest, *Mathematics, Education and Philosophy: An International Perspective*, 28.